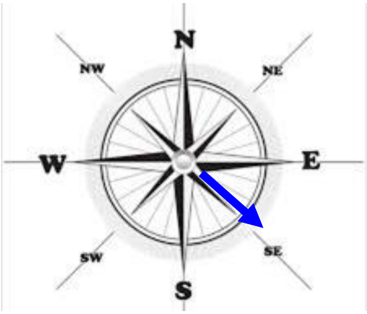
Dynamic Programming

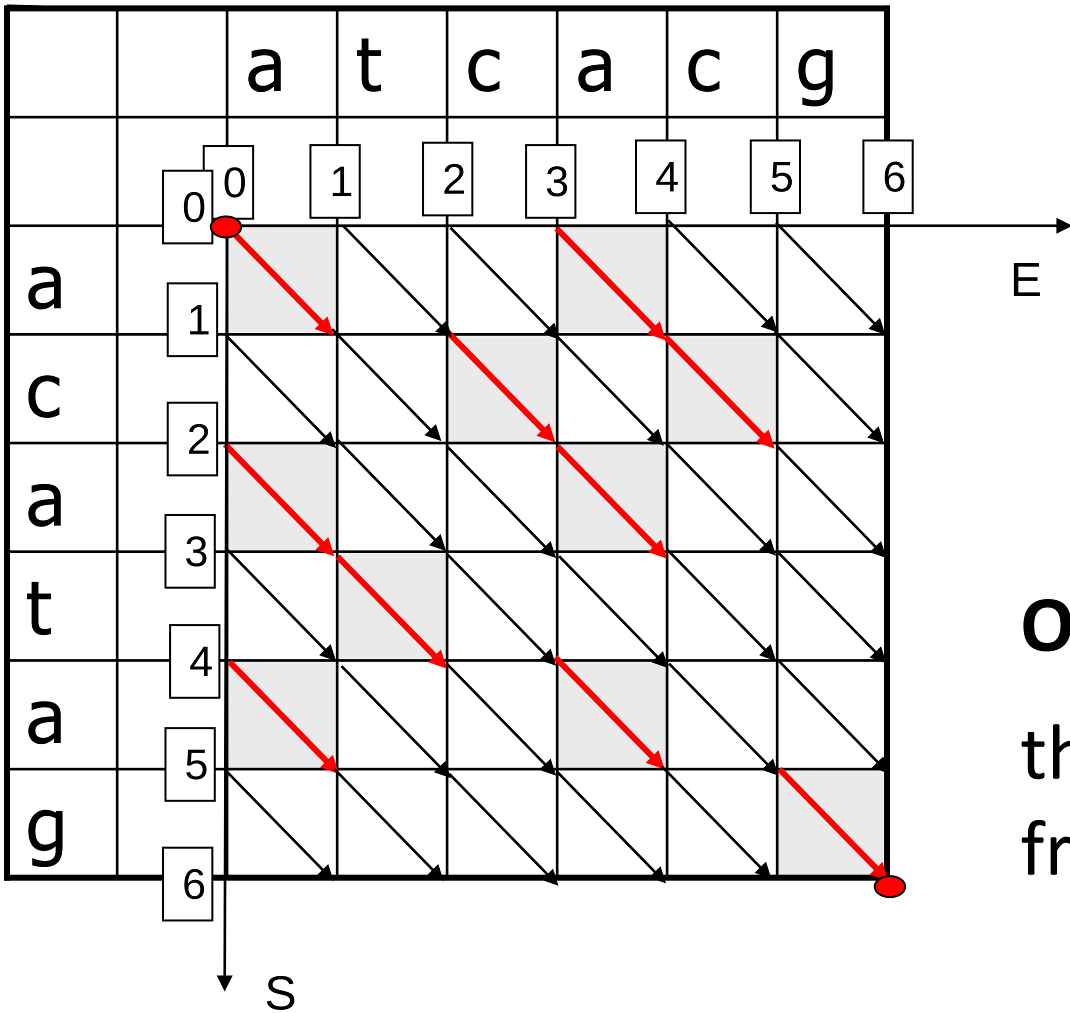
Lecture 07.01

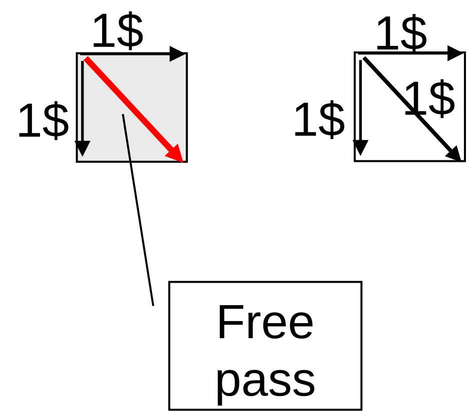
by Marina Barsky



Problem: the cheapest path in a special grid

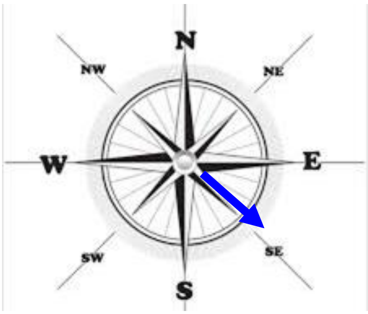
Input:



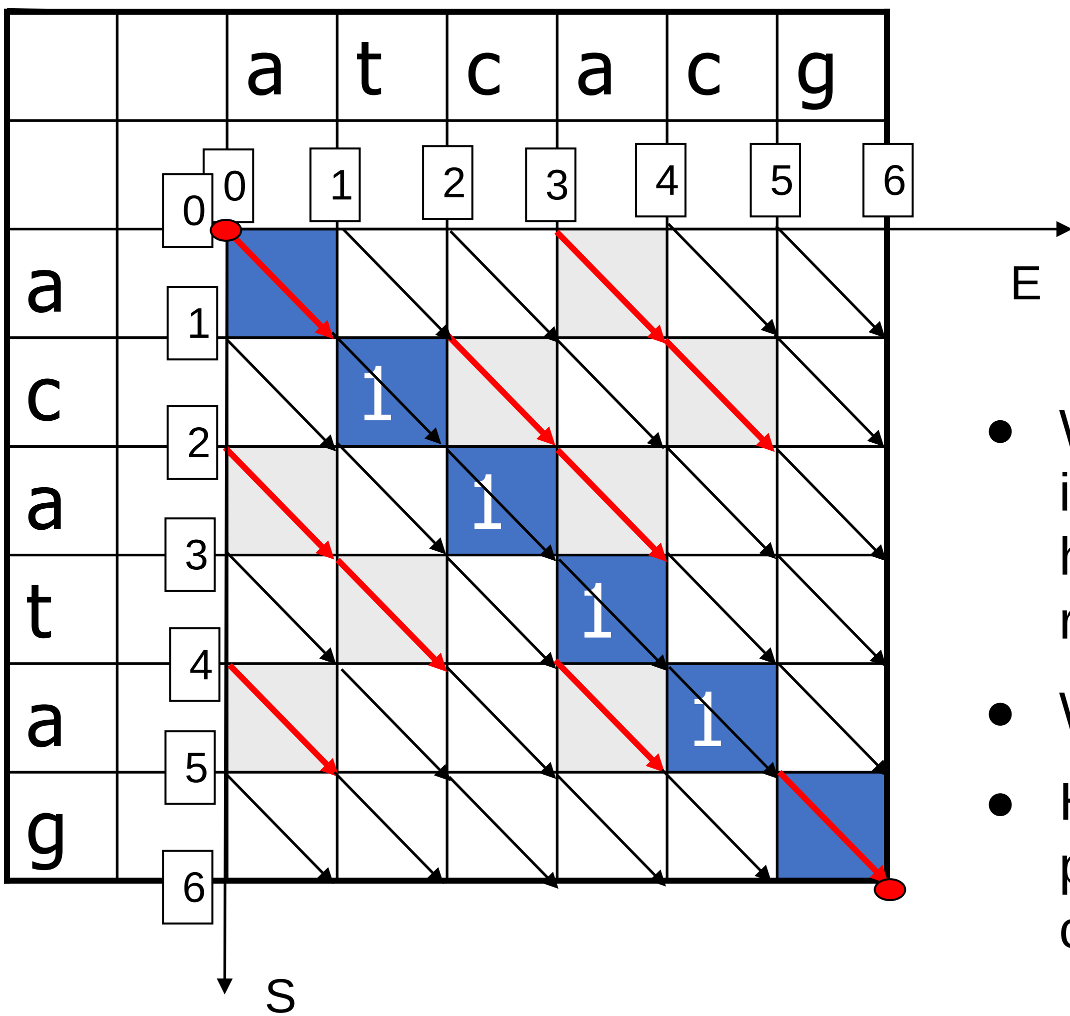


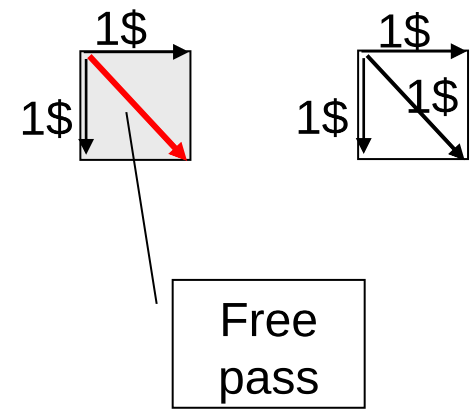
Output:

the cheapest path from (0,0) to (6,6)

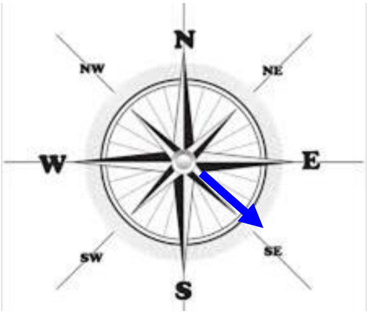


Without the map

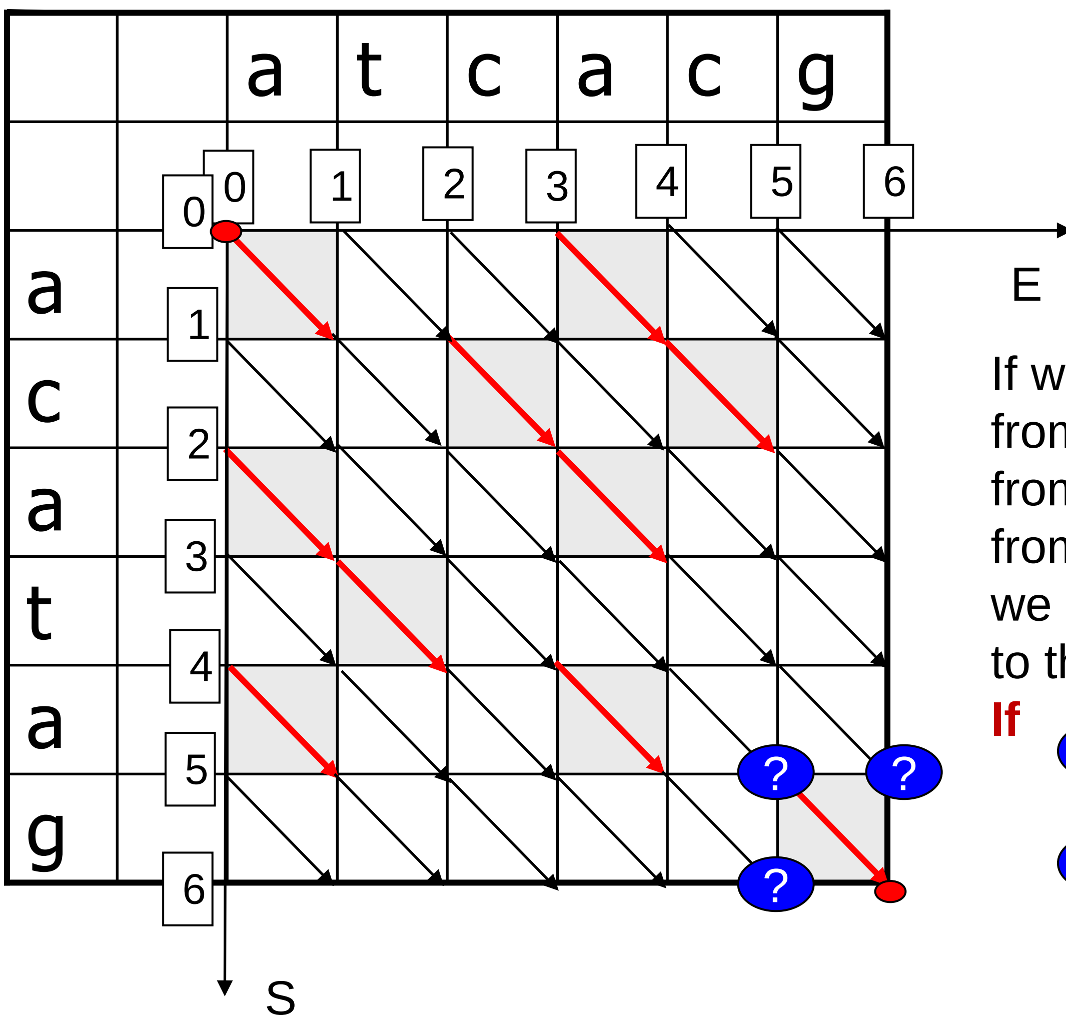


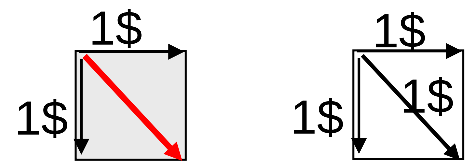


- Without additional information, we will always head South-East hoping to reach the destination faster
- We will pay 4$
- However a better (cheaper) path exists with more free cells



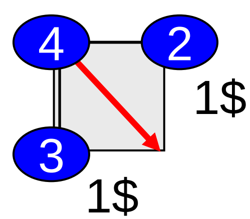
Sub-problems approach

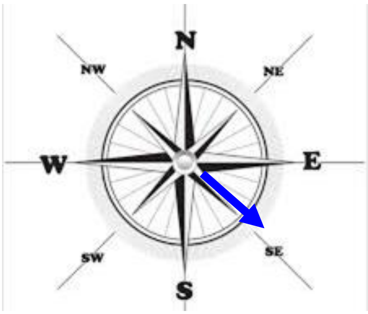




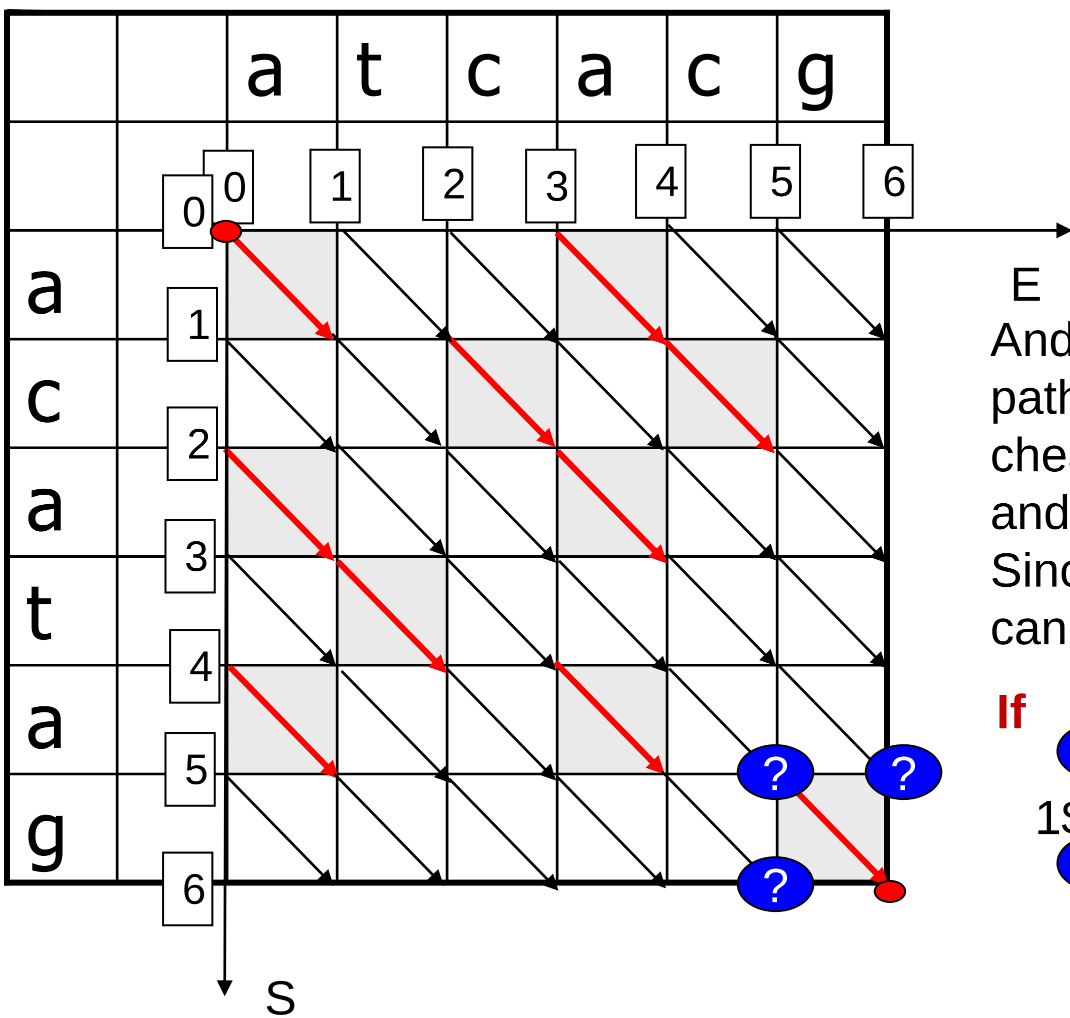
If we knew the cheapest paths
from (0,0) to (5,5)
from (0,0) to (6,5)
from (0,0) to (5,6)
we could choose the best last step
to the destination:

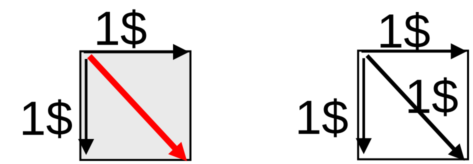
If





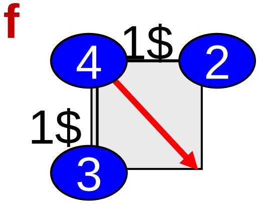
Sub-problems approach



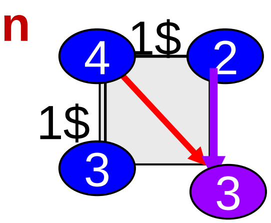


And this is true for any cell – what path to choose depends on the cheapest paths to the left, upper, and upper-left corner.
Since we choosing only 1 step, we can take the min of the result

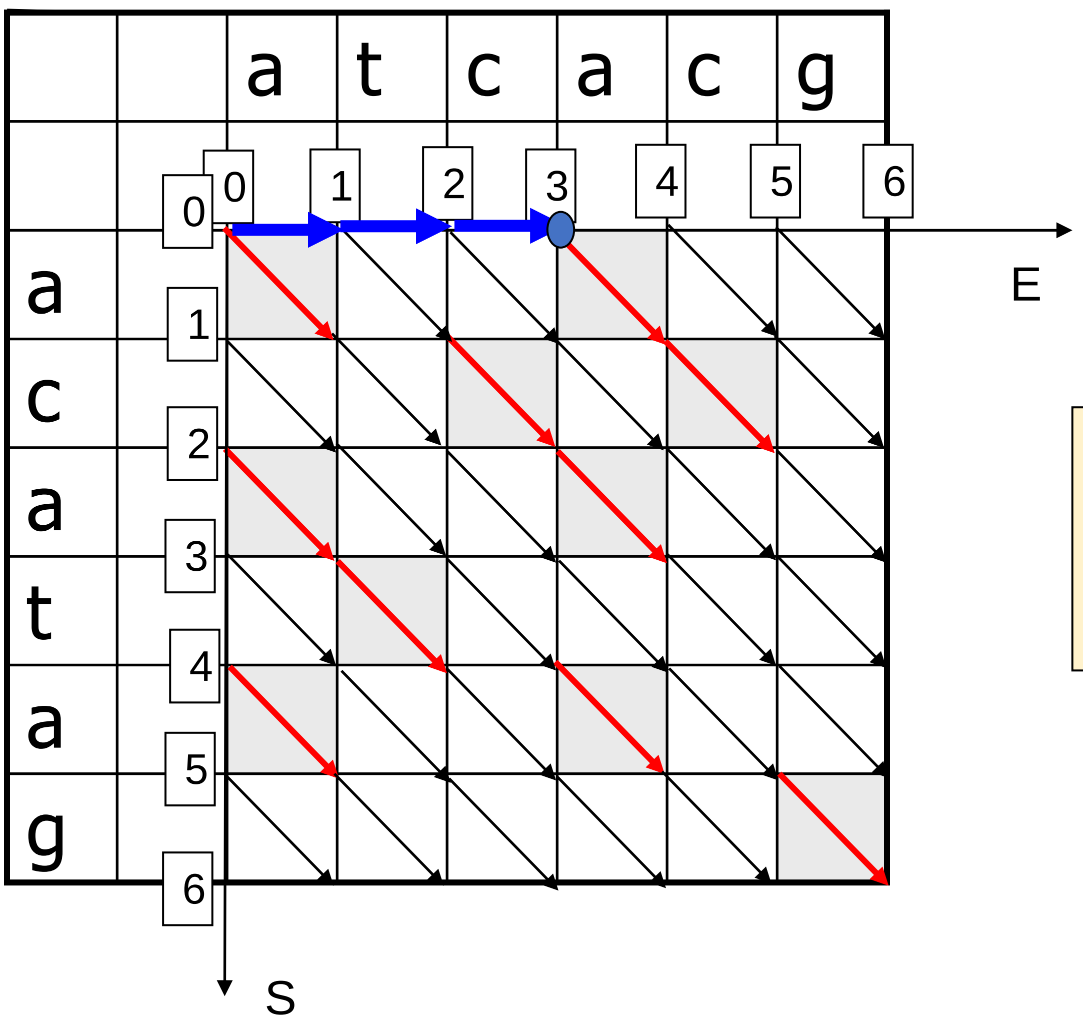
If



then



Recurrence relation – base condition



When i=0, there is no cheaper way of going from (0,0) to (0,j) than to pay j$ - heading strictly to the right (East)

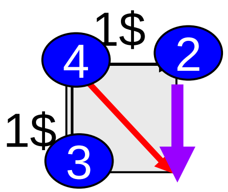
The same for j=0.

The base condition:

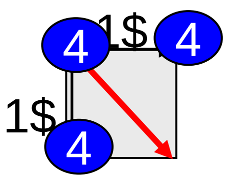
if i=0 then COST(i,j)=j

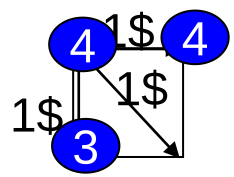
if j=0 then COST(i,j)=i

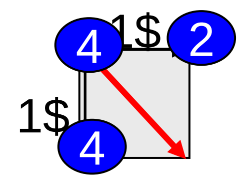
Recurrence relation (for i>0 and j>0)



$$\text{COST}(i,j) = \min \begin{cases} \text{COST}(i-1,j)+1 \\ \text{COST}(i,j-1)+1 \\ \text{COST}(i-1,j-1)+\text{DIAGONAL}(i,j) \end{cases}$$

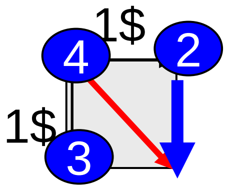






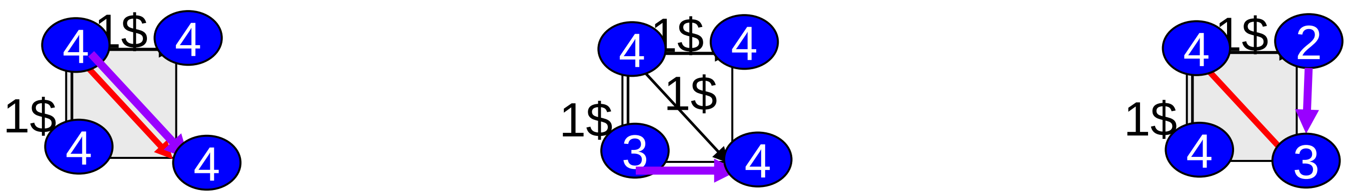
For each case, what is the best choice?

Recurrence relation (for i>0 and j>0)



$$COST(i,j) = \min \begin{cases} COST(i-1,j)+1 \\ COST(i,j-1)+1 \\ COST(i-1,j-1)+DIAGONAL(i,j) \end{cases}$$

For each case, what is **the best choice**?



Recursive algorithm

$$COST(i,j) = \min \begin{cases} COST(i-1,j)+1 \\ COST(i,j-1)+1 \\ COST(i-1,j-1)+DIAGONAL(i,j) \end{cases}$$

algorithm cheapestPath (**array** *diagonalCost, N*, *M*)

return *cost* (*N*, *M, diagonalCost*)

algorithm *cost* (*i*, *j, diagonalCost*)

if *i*=0 **then**

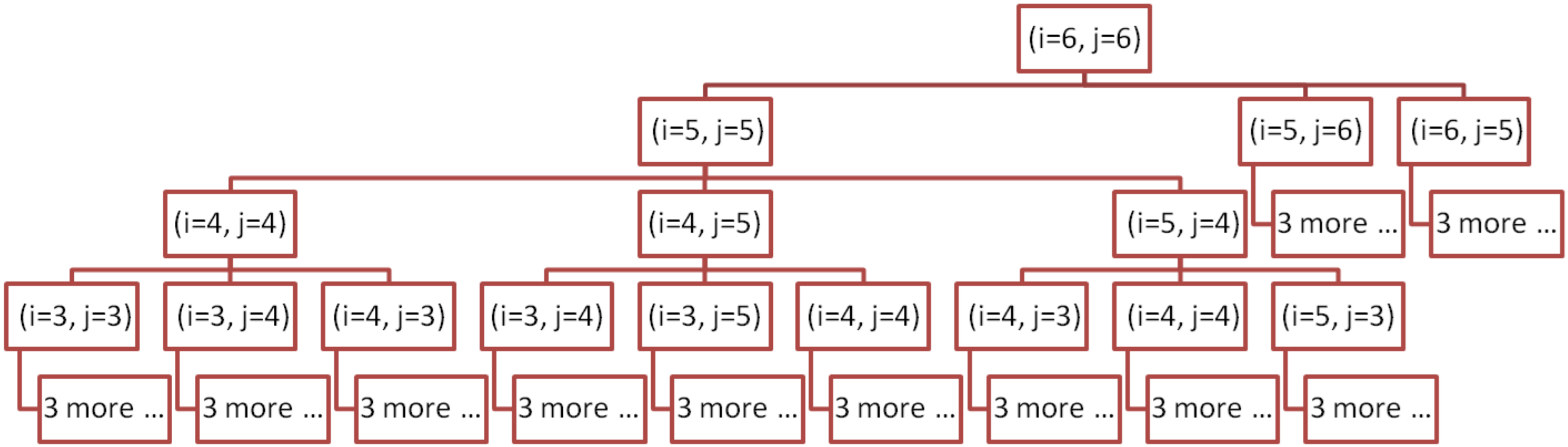
return *j*

if *j*=0 **then**

return *i*

return min (***cost*** (*i*-1, *j*) +1, ***cost*** (*i*, *j*-1)+1, ***cost*** (*i*-1, *j*-1)+*diagonalCost* [*i*] [*j*])

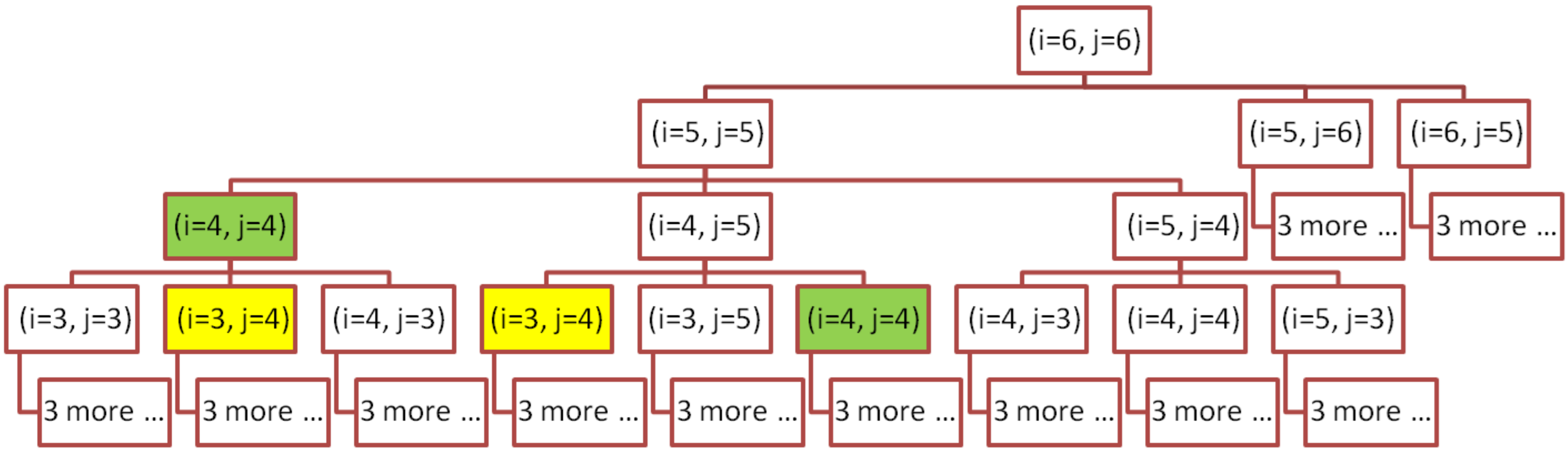
The recursion tree: $O(3^N)$



$O(3^N)$?

But there are only N*M different combinations (*i*,*j*)!

Recursive algorithm: $O(3^N)$



The algorithm is exponential in N because **we call the recursive function multiple times with the same parameters!**

Idea 1: store intermediate results

- Store the results of the *cost(i,j)* in a 2D table – so they do not need to be recomputed when needed again
- There are at most N^2 different combinations of (*i,j*)
- For each combination of (i,j) we compute the cost(i,j) only once
- When we need cost(i,j) again, we first check if it is already computed
- This gives a total running time $O(N^2)$

- The method of storing the results of recursive calls in a lookup table is called ***recursion with memoization***

Idea 2: The bottom-up computation

- In this particular problem we would need to compute the cost for all combinations of (*i, j*)
- Hence, instead of starting from *cost*(*N,M*) - fill in the best values for each cell of *N*M* table **starting from the lowest values**

The bottom-up computation

- Create a table of size(NxM) to store results of cost(i, j) for each $0 \leq i \leq N$ and $0 \leq j \leq M$
- First, fill-in the basic values of recursion – for i=0 and for j=0
- Apply recursive formula for computing the value of each cell from the lowest numbers of i and j to the highest (by rows or by columns)

- At the end, we will have the cost of the best path in the cell (N, M)

The recurrence relation: stays the same

The base condition:

if i=0 then COST(i,j)=j
if j=0 then COST(i,j)=i

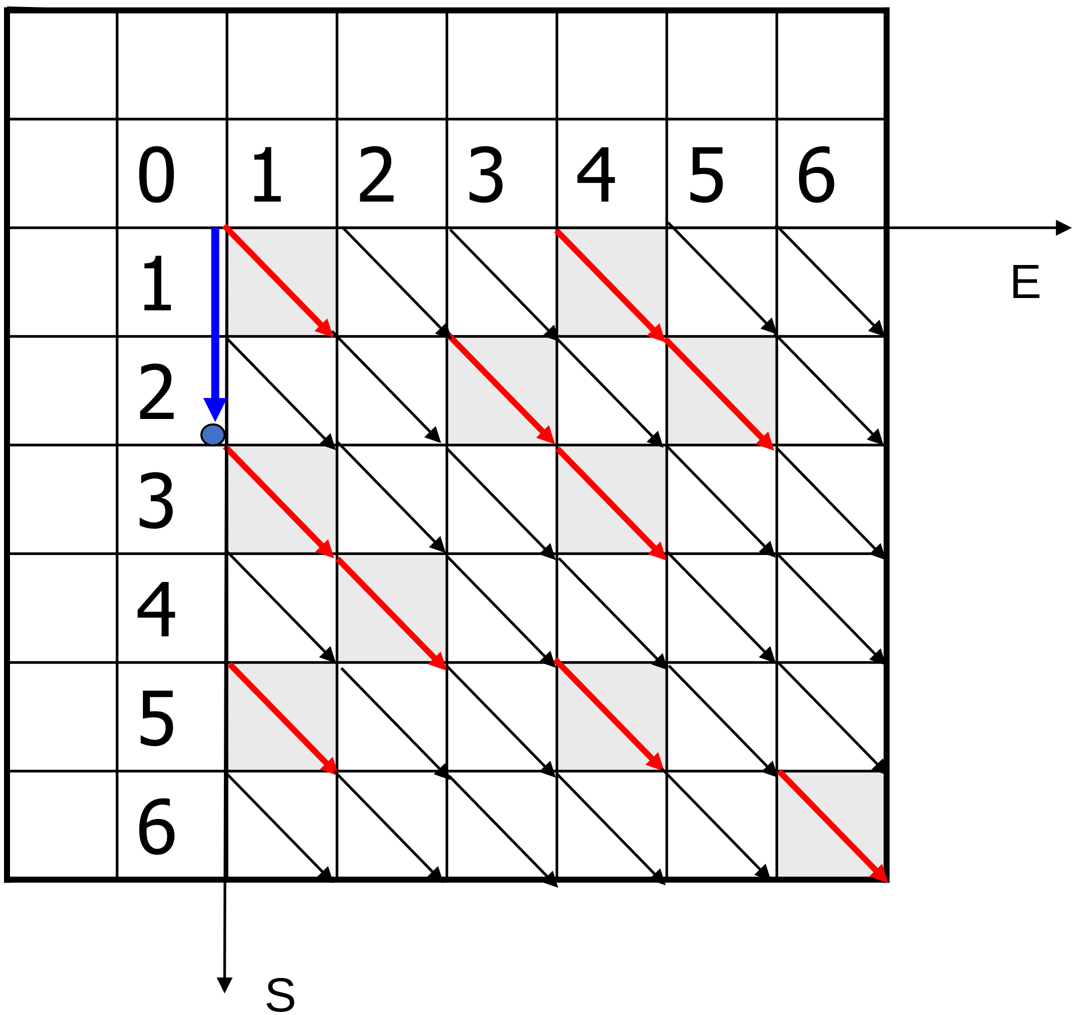
The main relation (for i>0 and j>0)

$$COST(i,j)=\min \begin{cases} COST(i-1,j)+1 \\ COST(i,j-1)+1 \\ COST(i-1,j-1)+DIAGONAL(i,j) \end{cases}$$

We change:

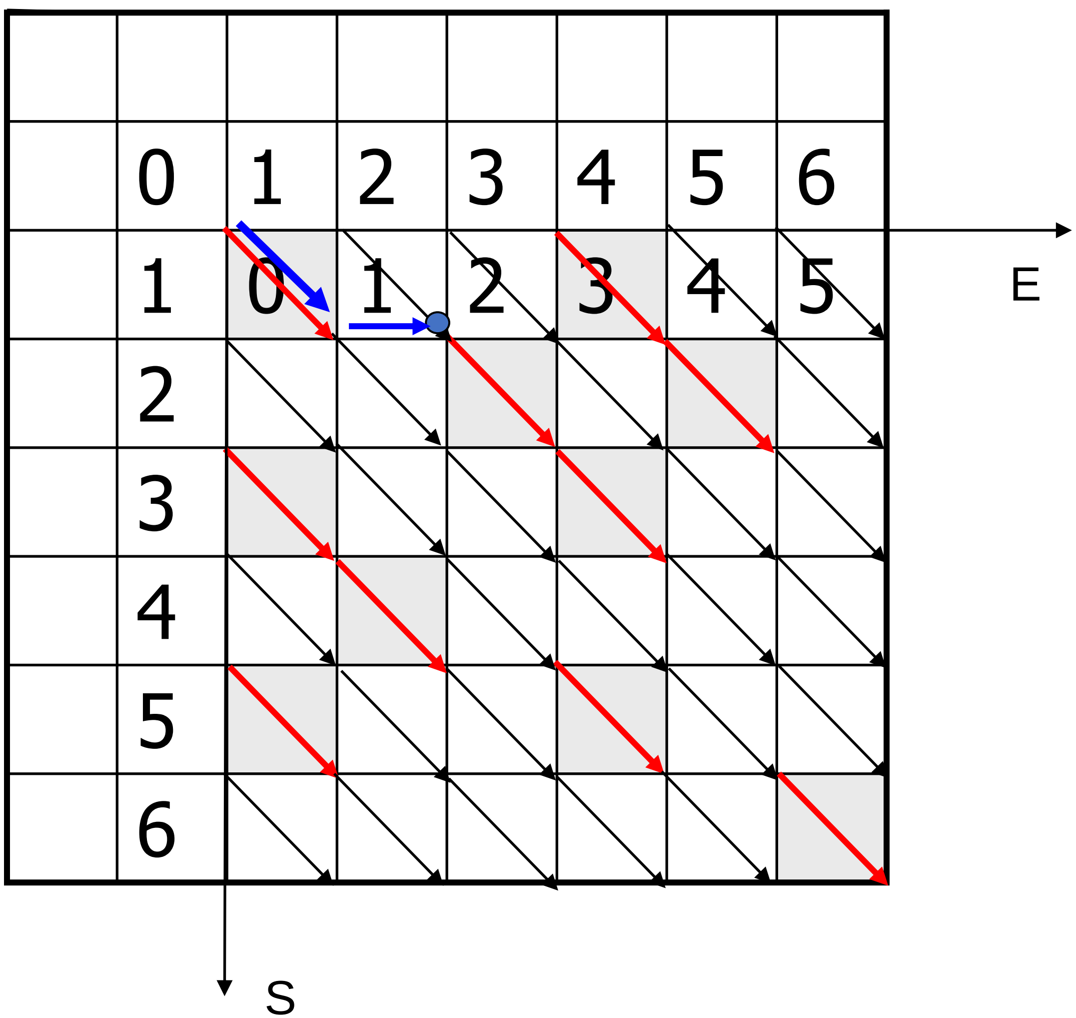
the order of computation

Fill values for i=0 and for j=0 (the base case)



There is no cheaper way of going to the point (2,0) than paying 2 $

Fill values for i=1 (from left to right)



Cell(1,2)=1

since the cheapest possible way is to continue the free path through the cell (1,1)

Fill the entire table (left-to-right top-down)

	0	1	2	3	4	5	6
	1	0	1	2	3	4	5
	2	1	1	1	2	3	4
	3	2	2	2	1	2	3
	4	3	2	3	2	2	3
	5	4	3	3	3	3	3
	6	5	4	4	4	4	3

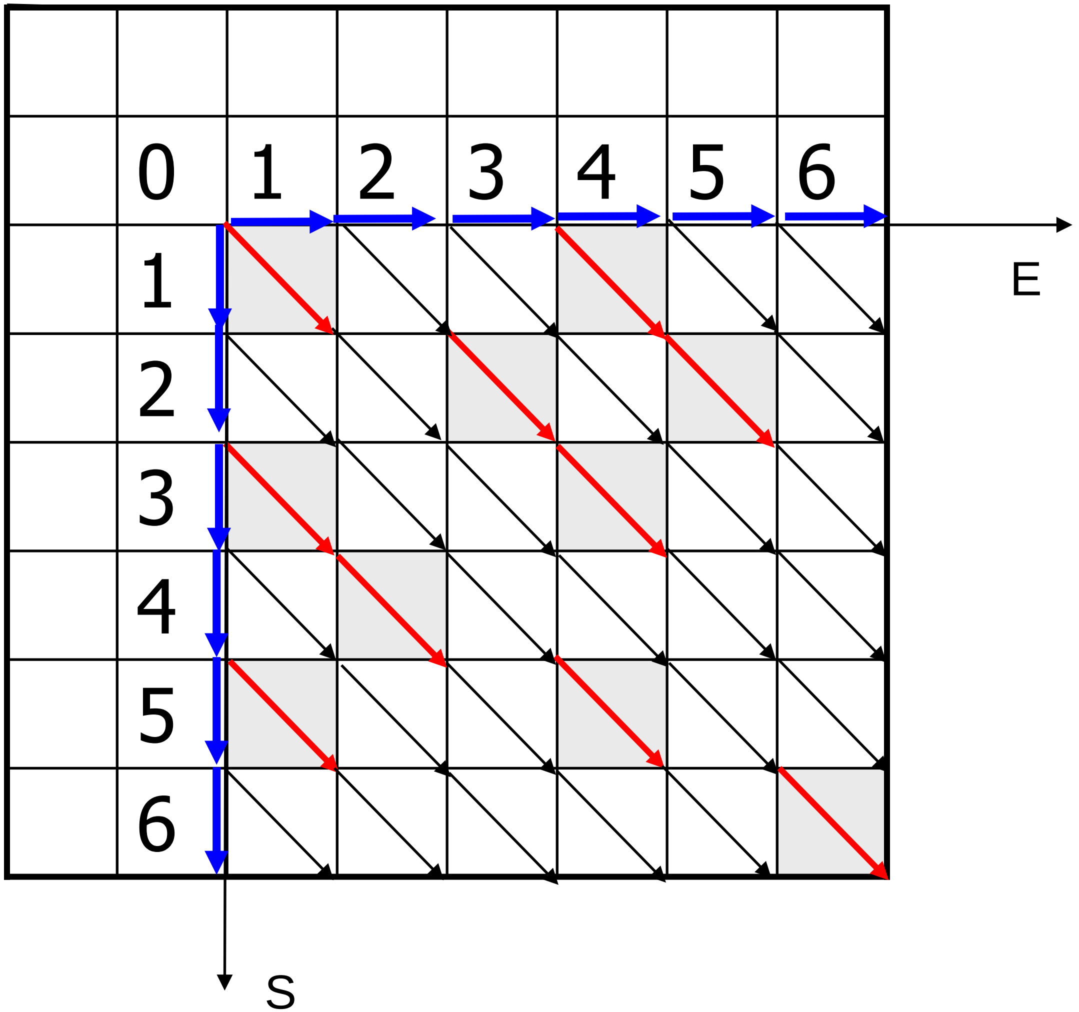
E

S

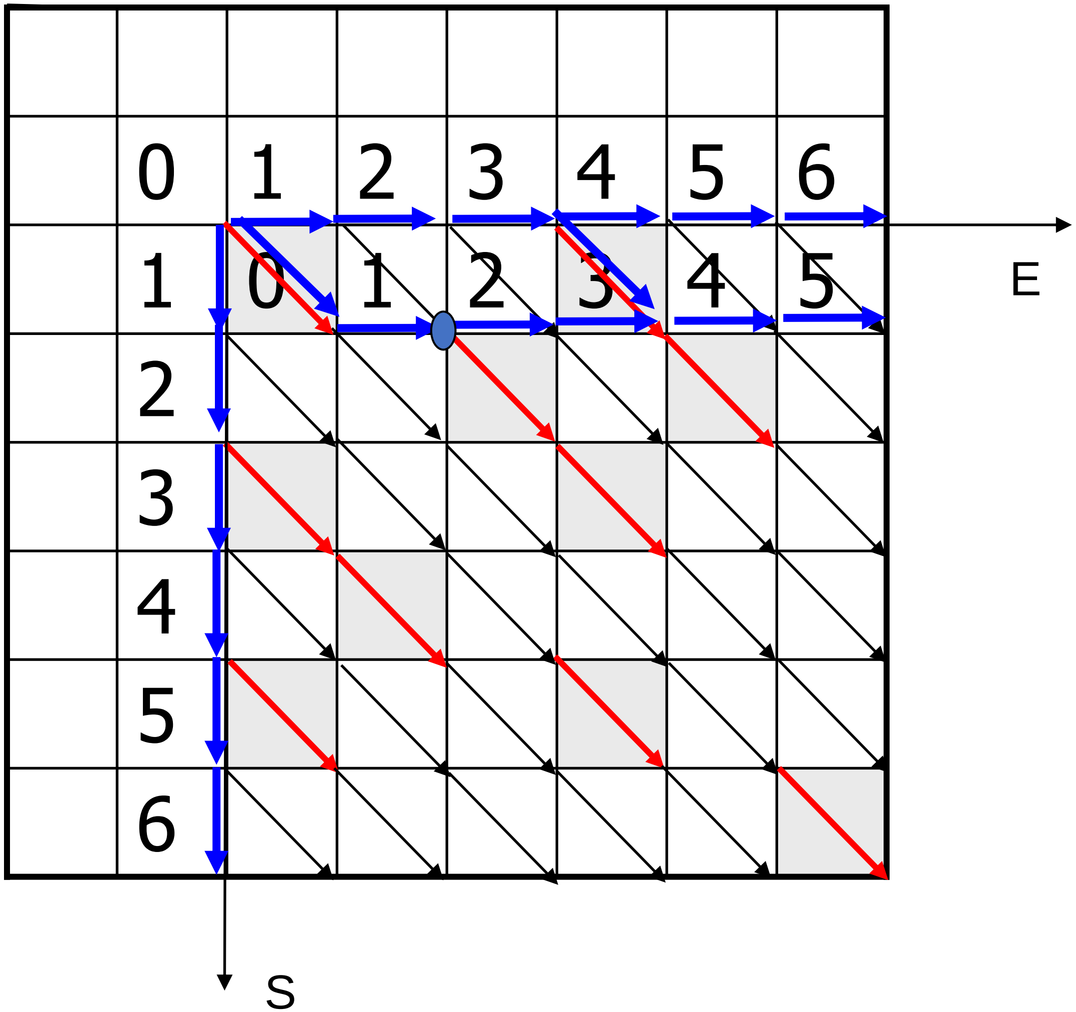
The overall cheapest possible path costs 3$

But what is this path?

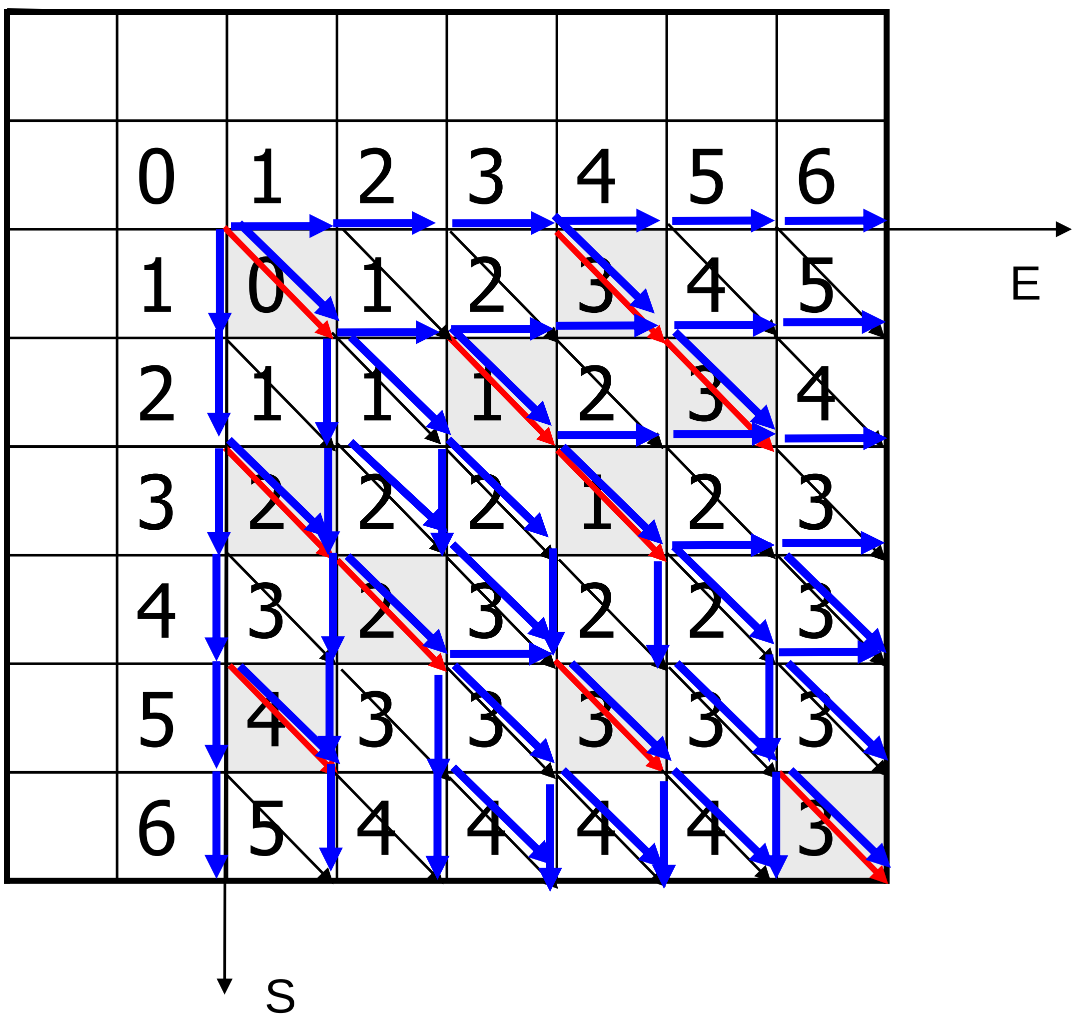
Keeping track of the source



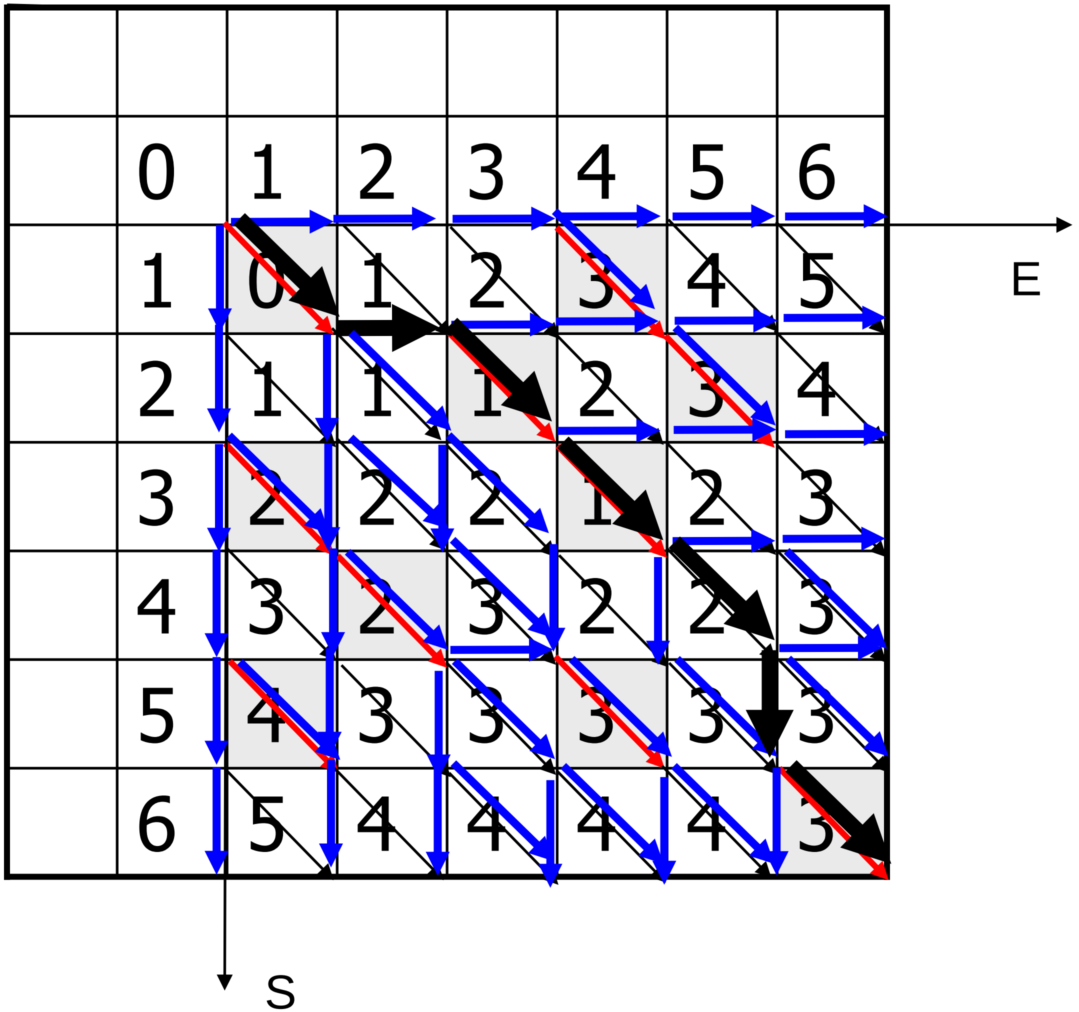
Keeping track of the source



Keeping track of the source



Trace back –
how did we get the path with the cost 3?



Our first *Dynamic Programming* algorithm

Algorithm: cheapestPath (diagonalCost NxM)

```
allocate array DPTable (NxM)
DPTable [0][0]:=0

for i from 1 to N:
    DPTable [i][0]:=i
for j from 1 to M:
    DPTable [0][j]:=j

for i from 1 to N:
    for j from 1 to M:
        DPtable [i][j]:=min (DPtable [i-1][j-1]+ diagonalCost [i][j],
                                          DPtable [i-1][j]+1, DPtable [i][j-1]+ 1)

return DPTable [N][M]
```

2 nested loops: $O(N^2)$

Dynamic programming: when

- We want to optimize something: min, max
- The solution to the problem depends on the solutions to subproblems
- We would need the solutions to all subproblems
- Subproblems overlap

Dynamic programming: how

- The recurrence relation
- The bottom-up computation
- The traceback

"Programming" in "Dynamic programming" has nothing to do with programming!

- Richard Bellman developed this idea in 1950s working on an Air Force project
- At that time, his approach seemed completely impractical
- He wanted to hide that he is really doing pure math from the Secretary of Defense

Richard Bellman

. . . What name could I choose? I was interested in planning but *planning* is not a good word for various reasons. I decided therefore to use the word "programming" and I wanted to get across the idea that this was dynamic. **It was something not even a Congressman could object to.** So I used it as an umbrella for my activities.

Representative problems

- ❑ Edit distance
- ❑ Knapsack 01
- ❑ Shortest paths

Edit distance

Transforming one sequence into another: *edit operations*

- We can transform the first string S1 into the second S2 by applying a sequence of edit operations on S1 :
  - Deleting 1 symbol
  - Inserting 1 symbol
  - Replacing 1 symbol

S1	a	c	t			a	t	g
		Delete c		Insert a	Insert c		Delete t	
S2	a		t	a	c	a		g

In total, 4 edit operations

String alignment

- An *alignment* of 2 strings is obtained by first inserting spaces (gaps), either into or at the end of both strings, and then placing 2 resulting strings one above the other, so that every character or space in either string is opposite a single character or space in the other string

Alignment

S1	a	c	t	-	-	a	t	g
S2	a	-	t	a	c	a	-	g

4 gaps,

no mismatches

Edit distance: definition

- The ***edit distance*** between two strings is defined as the **minimum** number of edit operations needed to transform one string into another

S1	a	c	t	a	t		g
S2	a		t	a	c	a	g

Delete c

Replace t

Insert a

In total, 3 edit operations

Optimal alignment

- An optimal alignment is obtained from the calculation of the edit distance

Optimal Alignment

S1	a	c	t	a	t		g
S2	a		t	a	c	a	g

Delete c

Replace t

Insert a

Edit distance=3

Is this really the smallest number of edit operations?

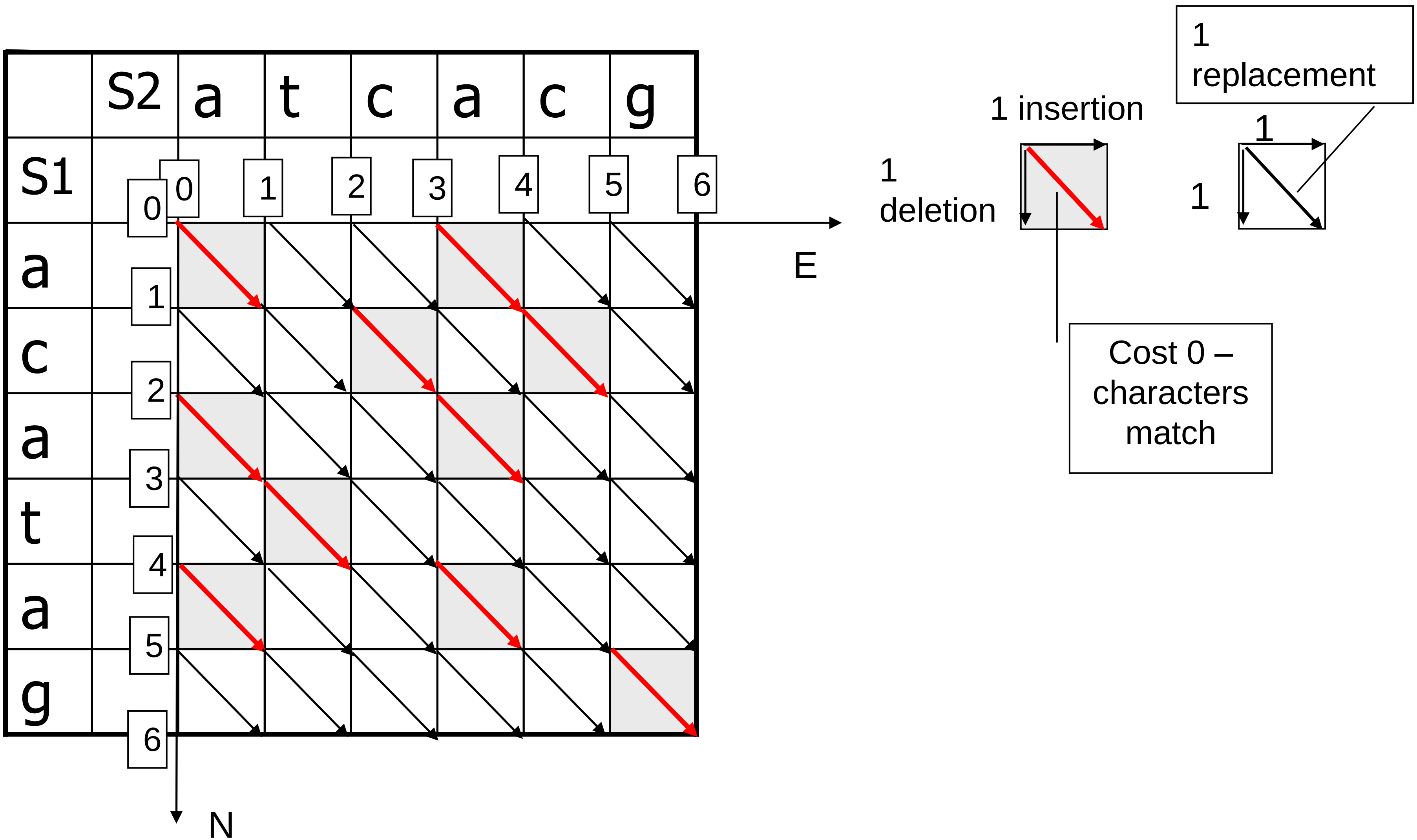
How do we compute edit distance in general?

The edit distance problem

Input: 2 strings S_1 and S_2

Output: the *edit distance* between two strings along with a sequence of the operations which describe the transformation

Full analogy with the cheapest path

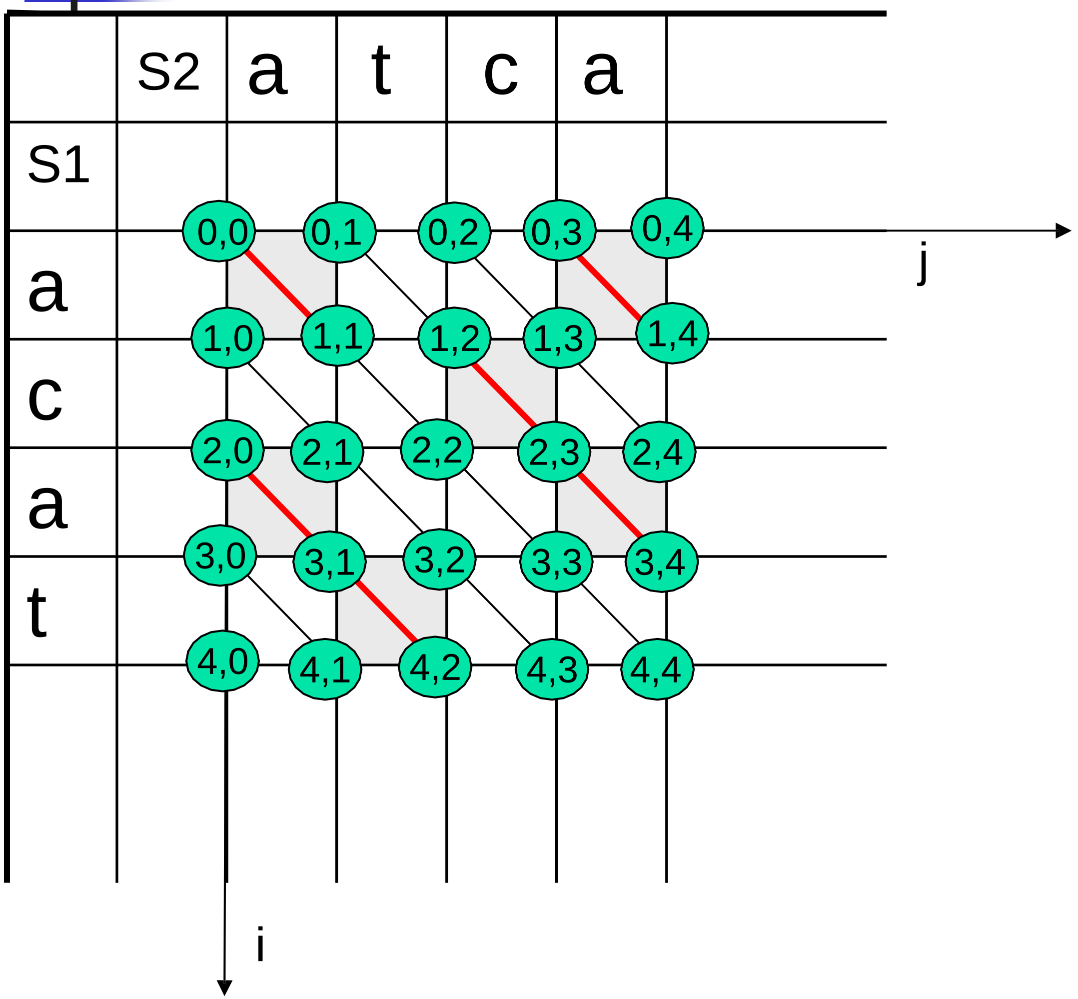


The dynamic programming solution to the edit distance problem

Trivially follows from the solution for the cheapest path:

- If we moved strictly down in the grid, we deleted (ignored) 1 symbol from S1
- If we moved strictly to the right, we inserted 1 symbol from S2 into S1
- If we moved by diagonal of cost 0, we matched the corresponding characters
- If we moved by diagonal of cost 1, we replaced one symbol in S1 with the corresponding symbol in S2

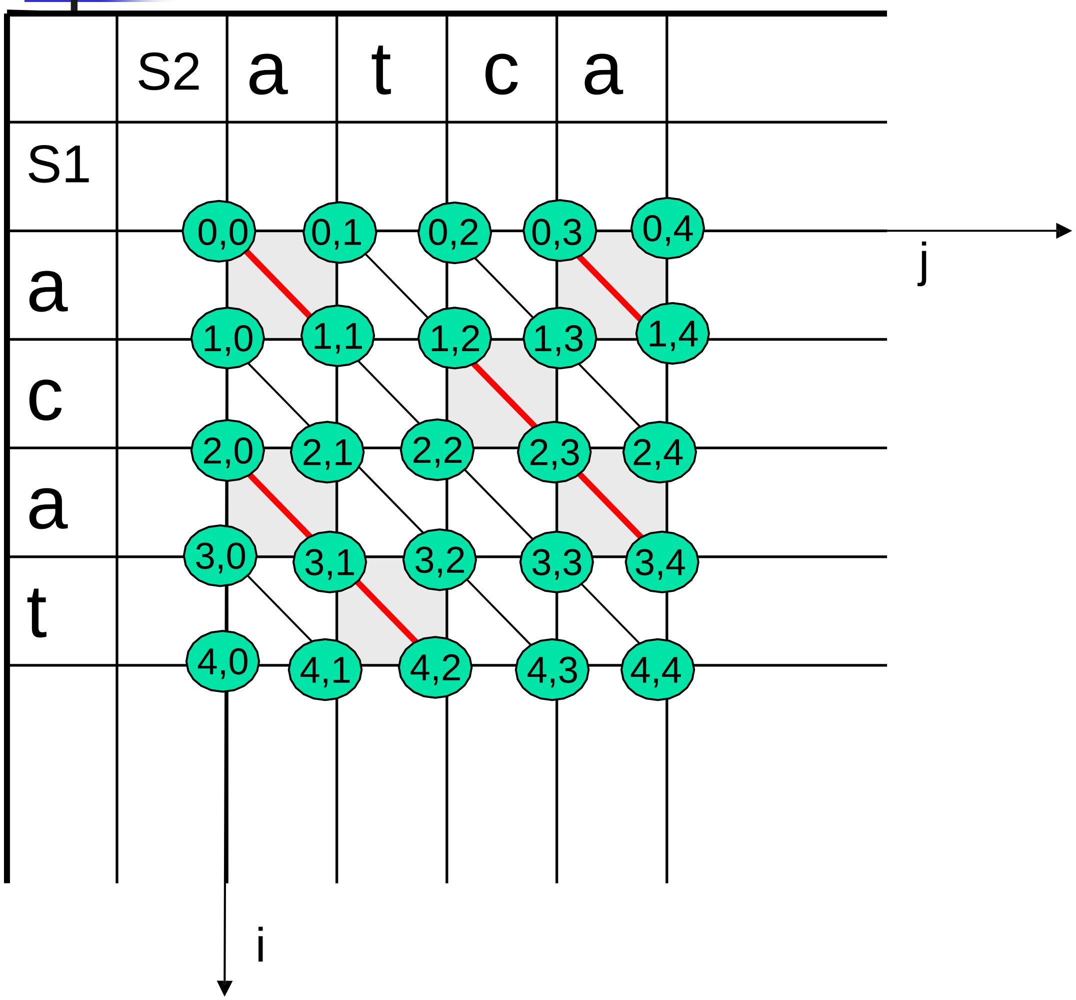
Useful abstraction: *edit graph*



An ***edit graph*** for a pair of strings S_1 and S_2 has $(N+1)*(M+1)$ vertices, each labeled with a corresponding pair (i,j), $0 \le i \le N$, $0 \le j \le M$

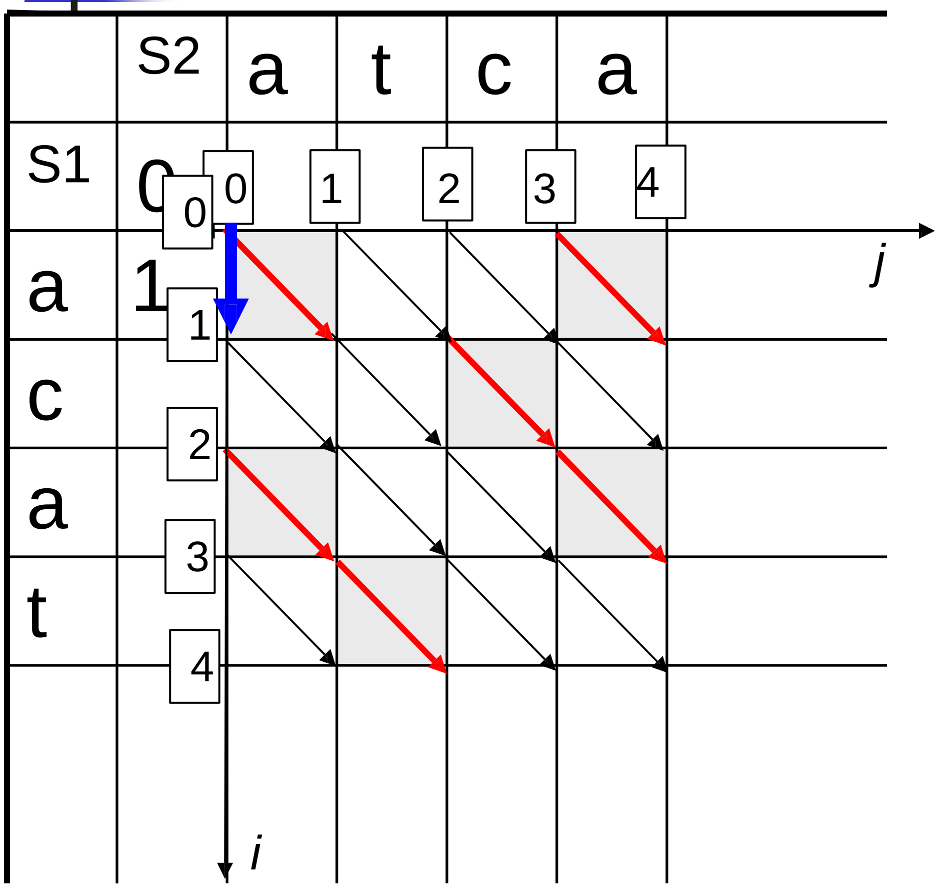
The edges are **directed** and their weight depends on the specific string problem: for the edit distance problem – red edges have cost 0, black edges have cost 1

The cheapest path in the edit graph



The cost of a **cheapest path** from vertex (0,0) to vertex (*N*,*M*) in this edit graph corresponds to the **edit distance** between S1 and S2, and the path itself represents a series of edit operations and an optimal alignment of S1 with S2

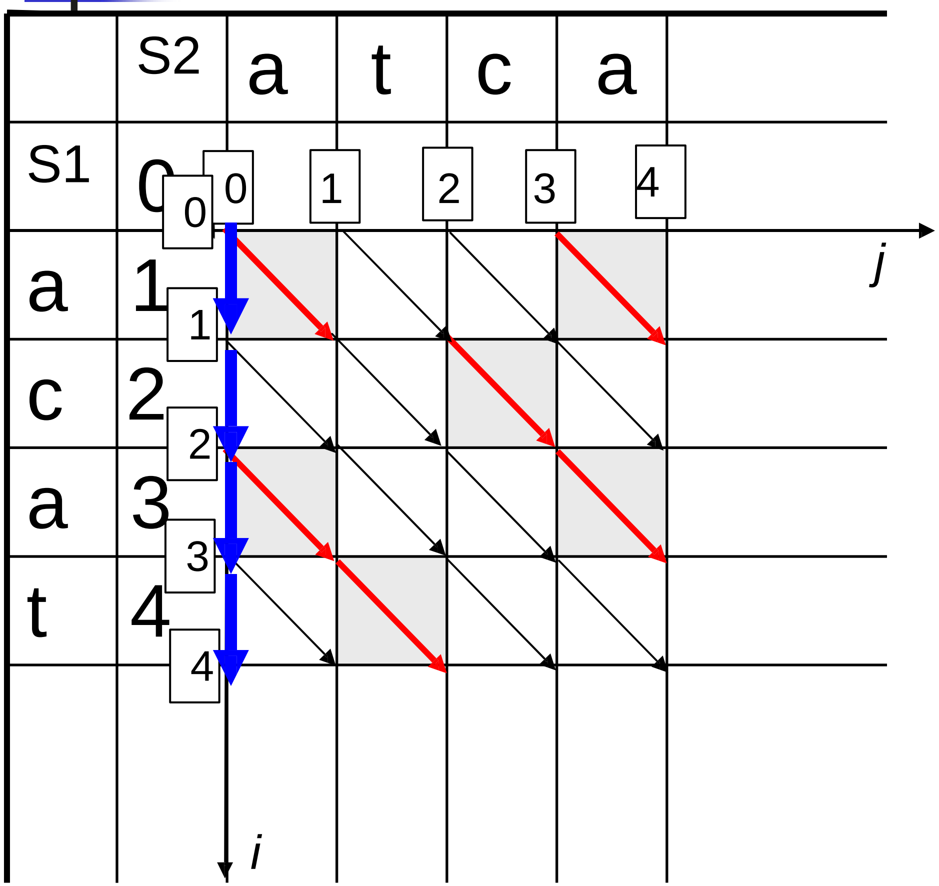
Calculating edit distance. Base condition



The minimum number of edit operations we need in order to transform string *a* into an empty string (of length 0) is 1 (deletion)

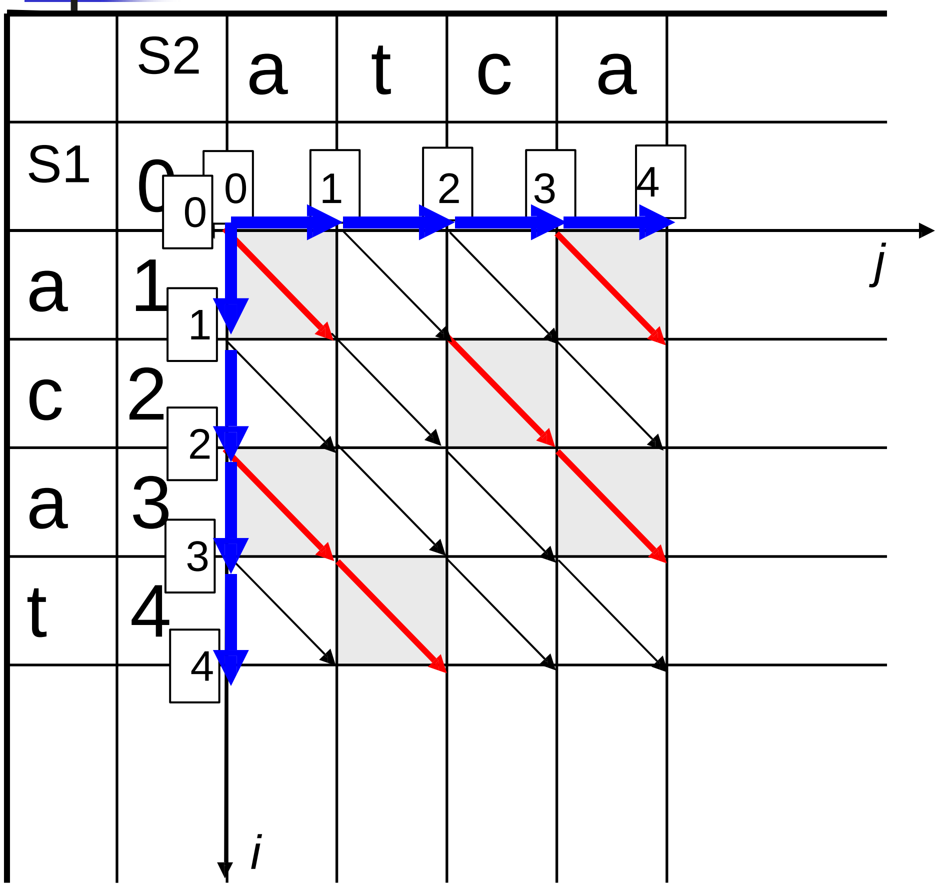
Therefore the minimum edit distance between ε and *a* is 1

Calculating edit distance. Base condition



The same is true for ε and *ac, aca, acat*

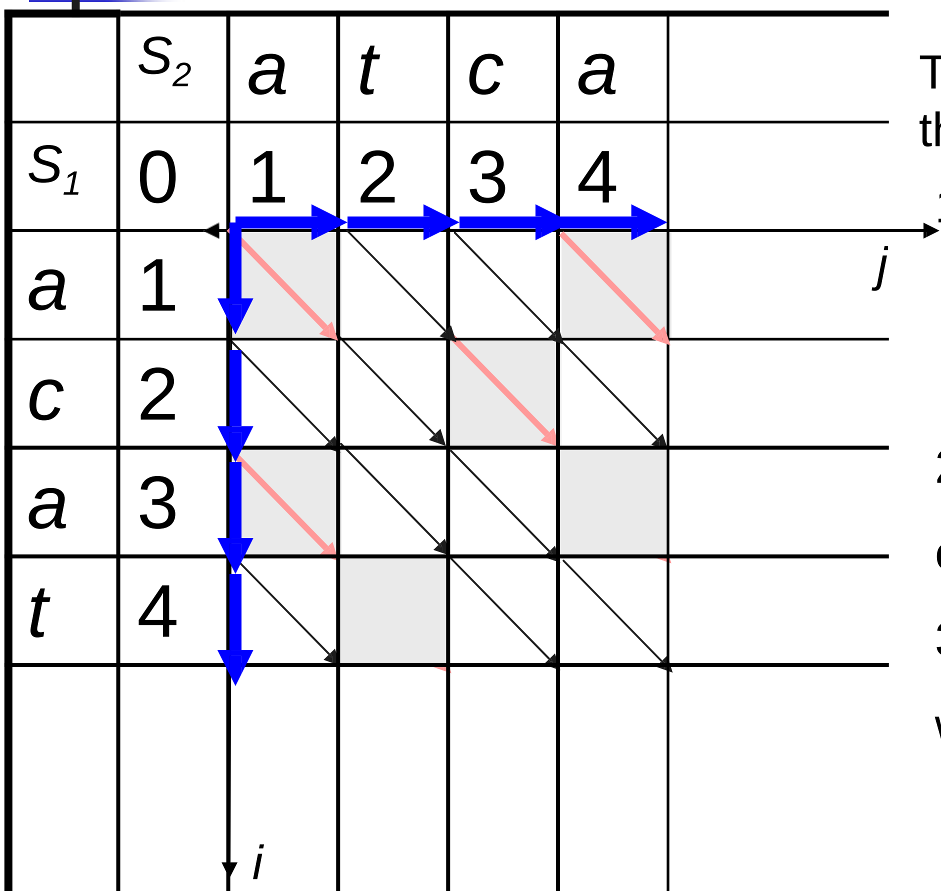
Calculating edit distance. Base condition



In order to transform ε into *a*, we need to insert 1 character. This is the best way to do it, there is no cheaper way.

The same for transforming ε into *at, atc, atca* with 2, 3, 4 insertions respectively

Calculating edit distance. Recursion for i>0 and j>0



There are only 3 different ways to move through the next cell in the graph:

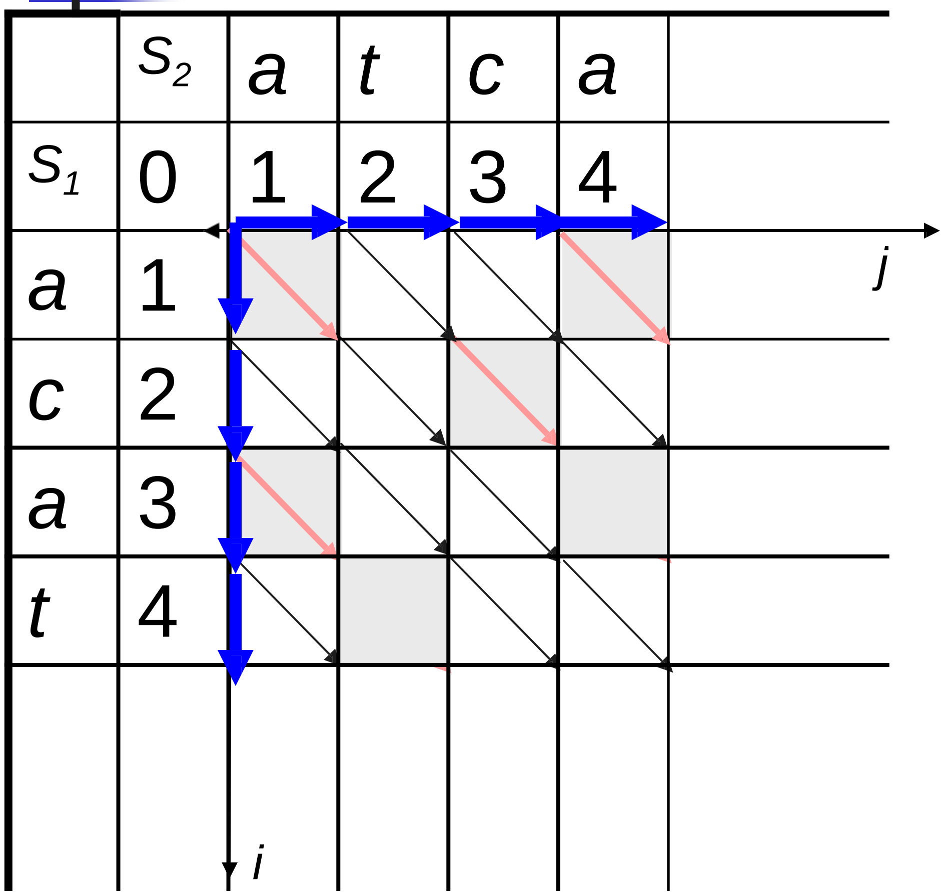
1. Increase both *i* and *j* (diagonal)

   if S1[i]≠S2[j] : 1 edit

   if S1[i]=S2[j] : 0 edits

2. Increase only *i* (insert $S_1[i]$) with the cost 1
3. Increase only *j* (delete - ignore $S_2[j]$) with the cost 1

Calculating edit distance. Recursion for i>0 and j>0



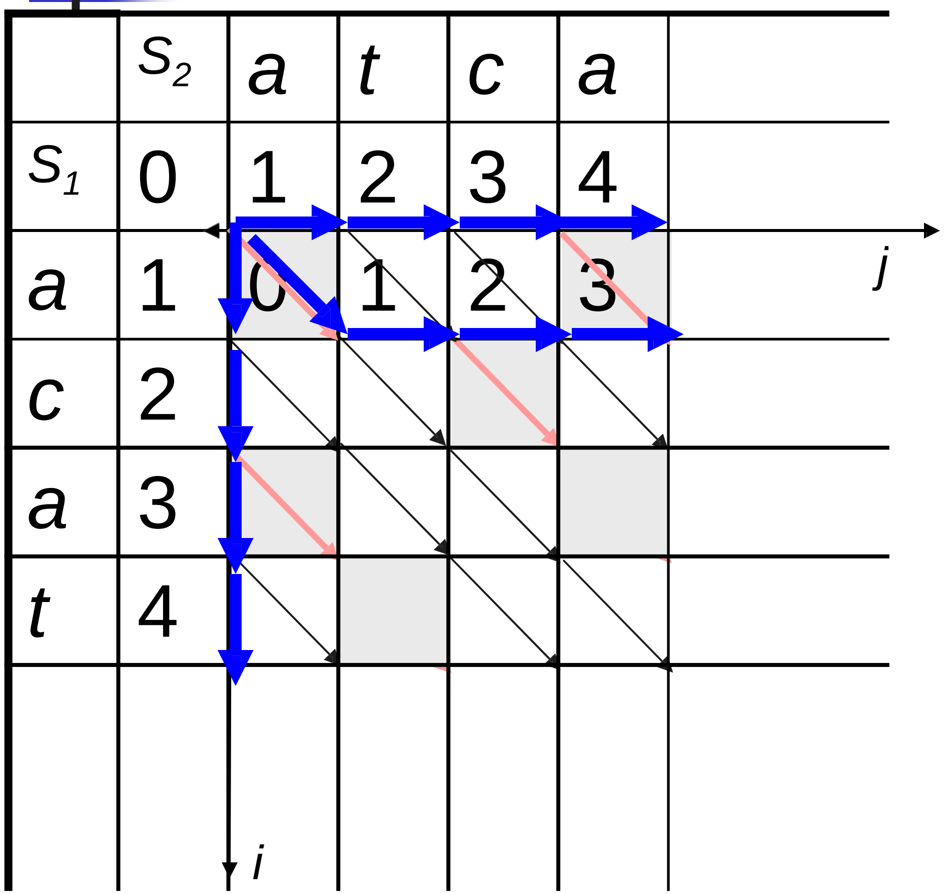
Thus, if we know the edit distance

D[*i*-1,*j*-1], *D*[*i*-1,*j*] and *D*[*i*,*j*-1], we can correctly calculate *D*[*i*,*j*]

This is true since there are no other ways of moving through cell [*i*][*j*].

Reaching the top, left and top-left corners by different paths cannot produce a better value than is already in these 3 cells, since they contain the minimum cost by definition

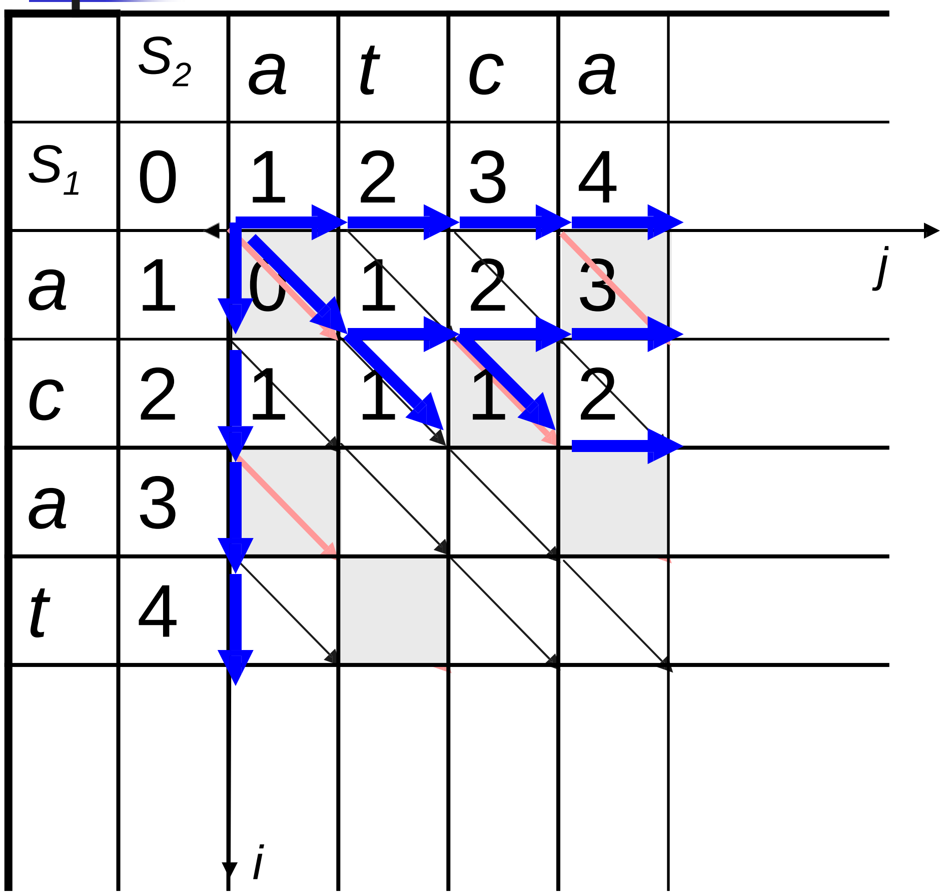
Calculating edit distance. Recursion for i>0 and j>0



$$D(i,j) = \min \begin{cases} D(i-1,j)+1 \\ D(i,j-1)+1 \\ D(i-1,j-1)+c(i,j) \end{cases}$$

$$\text{where } c(i,j) = \begin{cases} 0 \text{ if } S1[i]=S2[j] \\ 1 \text{ if } S1[i] \neq S2[j] \end{cases}$$

Calculating edit distance. Recursion for i>0 and j>0



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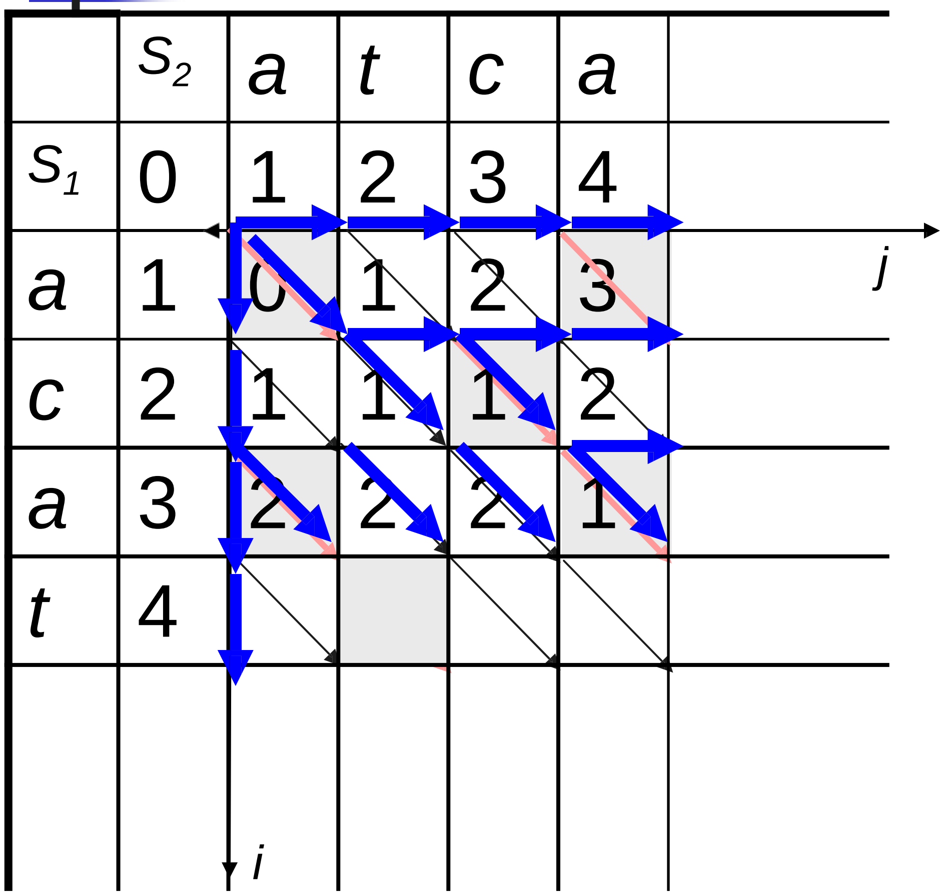
Calculating edit distance. Recursion for i>0 and j>0



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Calculating edit distance. Recursion for i>0 and j>0

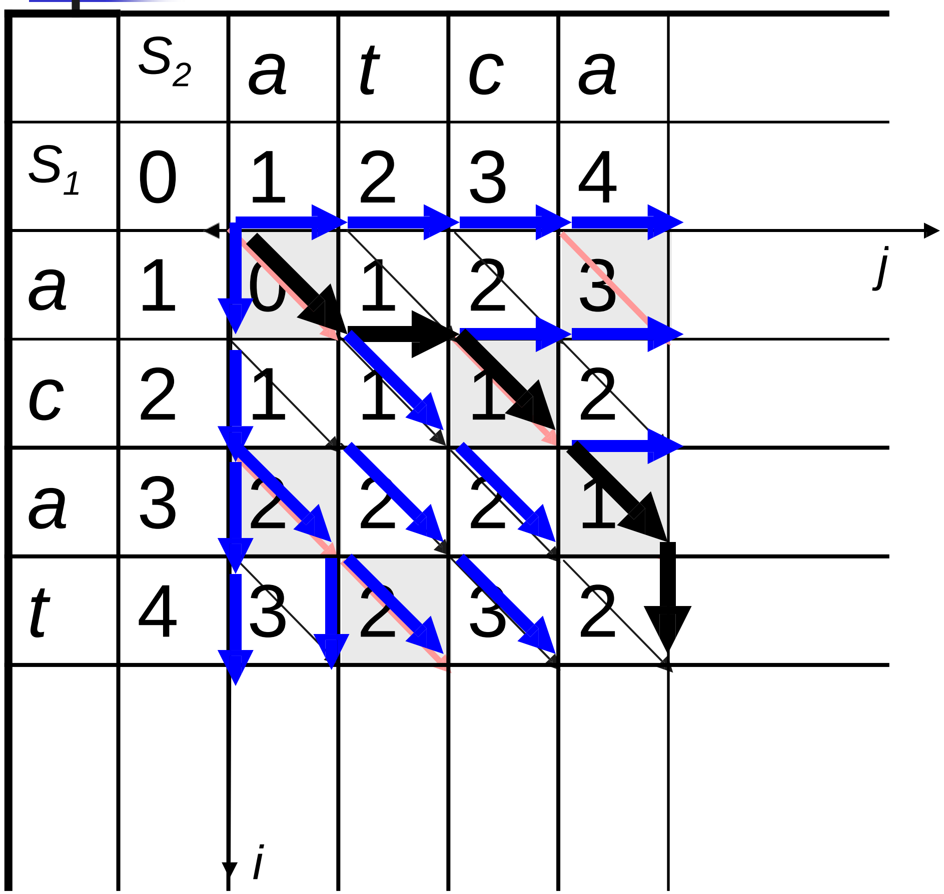
	S_2	*a*	*t*	*c*	*a*
S_1	0	1	2	3	4
a	1	0	1	2	3
c	2	1	1	1	2
a	3	2	2	2	1
t	4	3	2	3	2

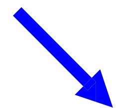
j *i*

$$D(i,j) = \min \begin{cases} D(i-1,j)+1 \\ D(i,j-1)+1 \\ D(i-1,j-1)+c(i,j) \end{cases}$$

$$\text{where } c(i,j) = \begin{cases} 0 \text{ if } S1[i]=S2[j] \\ 1 \text{ if } S1[i] \neq S2[j] \end{cases}$$

The sequence of edit operations





Place a character in S1 opposite to a character in S2

Place a character in S1 opposite to a gap in S2

Place a character in S2 opposite to a gap in S1

S1	*a*	-	*c*	*a*	*t*
S2	*a*	*t*	*c*	*a*	-

Optimal alignment

S1	*a*	*-*	*c*	*a*	*t*
S2	*a*	*t*	*c*	*a*	*-*

Explanation:

S_2 can be obtained from S_1 by a series of the following edit operations:

Insertion of *t* at position 2

Deletion of *t* at position 5

An optimal alignment is not unique

S1	*-*	*a*	*t*	*t*	*a*	*a*	*g*
S2	*t*	*a*	*-*	*t*	*c*	*a*	*g*

S1	*-*	*a*	*t*	*t*	*a*	*a*	*g*
S2	*t*	*a*	*t*	*c*	*a*	*-*	*g*

2 different alignments with the optimal edit distance 3